

# Role of skew-symmetric differential forms in mathematics

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## Abstract

Skew-symmetric forms possess unique capabilities. This is due to the fact that they deal with differentials and differential expressions; and, therefore, they are suitable for describing invariants and invariant structures. The properties of closed exterior and dual forms, namely, invariance, covariance, conjugacy and duality, either explicitly or implicitly appear in all invariant mathematical formalisms. The closed exterior forms relate to such branches of mathematics as algebra, geometry, differential geometry, vectorial and tensor calculus, the theory of complex variables, differential equations and so on. This enables one to see an internal connection between various branches of mathematics.

However, the theory of closed exterior forms cannot be completed without an answer to a question of how the closed exterior forms emerge. In the present paper we discuss essentially new skew-symmetric forms, which possess capabilities so far not exploited in any mathematical formalisms, namely, they generate closed exterior forms. Such skew-symmetric forms, which are evolutionary ones, are derived from differential equations, and, in contrast to exterior forms, they are defined on nonintegrable manifolds. Theory of evolutionary forms, what includes such elements as nonidentical relations, degenerate transformations, transition from nonintegrable manifold to integrable one, enables one to understand the process of conjugating operators, the mechanism of generation of invariant structures and others.

## 1 Exterior differential forms

In this Section introductory information on exterior differential forms is presented, and the basic properties of closed exterior differential forms and specific features of their mathematical apparatus are described.

The concept of “Exterior differential forms (differential forms with exterior multiplication)” was introduced by E.Cartan as a general concept for integrands, which constitute integral invariants [1]. (The existence of integral invariants was recognized previously by A. Poincare while studying the general equations of dynamics.)

The exterior differential form of degree  $p$  ( $p$ -form) can be written as [2–4]

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \quad 0 \leq p \leq n \quad (1.1)$$

Here  $a_{i_1 \dots i_p}$  are the functions of the variables  $x^{i_1}, x^{i_2}, \dots, x^{i_n}$ ,  $n$  is the dimension of the space,  $\wedge$  is the operator of exterior multiplication,  $dx^i, dx^i \wedge dx^j, dx^i \wedge dx^j \wedge$

$dx^k, \dots$  is the local basis which satisfies the condition of exterior multiplication:

$$\begin{aligned} dx^i \wedge dx^i &= 0 \\ dx^i \wedge dx^j &= -dx^j \wedge dx^i \quad i \neq j \end{aligned} \quad (1.2)$$

[Below, the symbol for summing,  $\sum$ , and the symbol for exterior multiplication,  $\wedge$ , will be omitted. Summation over repeated indices is implied.]

The differential of the exterior form  $\theta^p$  is expressed as

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} dx^{i_1} dx^{i_2} \dots dx^{i_p} \quad (1.3)$$

and is a differential form of degree  $(p+1)$ .

Let us consider some examples of the exterior differential form whose basis is the Euclidean space domains.

We consider a 3-dimensional space. In this case the differential forms of zero-, first- and second degree can be written as, [4]:

$$\begin{aligned} \theta^0 &= a \\ \theta^1 &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\ \theta^2 &= a_{12} dx^1 dx^2 + a_{23} dx^2 dx^3 + a_{31} dx^3 dx^1 \end{aligned}$$

With account for conditions (1.2), their differentials are the forms

$$\begin{aligned} d\theta^0 &= \frac{\partial a}{\partial x^1} dx^1 + \frac{\partial a}{\partial x^2} dx^2 + \frac{\partial a}{\partial x^3} dx^3 \\ d\theta^1 &= \left( \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) dx^1 dx^2 + \left( \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right) dx^2 dx^3 + \left( \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1} \right) dx^3 dx^1 \\ d\theta^2 &= \left( \frac{\partial a_{23}}{\partial x^1} + \frac{\partial a_{31}}{\partial x^2} + \frac{\partial a_{12}}{\partial x^3} \right) dx^1 dx^2 dx^3 \end{aligned}$$

One can see the following.

a) Any function is the form of zero degree. Its basis is a surface of zero dimension, namely, a variety of points. The differential of the form of zero degree is an ordinary differential of the function.

b) The form of first degree is a differential expression. As it was pointed out above, an ordinary differential of a function is an example of the first-degree form.

c) Coefficients of differentials of forms of zero-, first- and second degrees give the gradient, curl, and divergence, respectively. That is, the operator  $d$ , referred to as the exterior differentiation, is an abstract generalization of the ordinary operators of gradient, curl, and divergence. At this point it should be emphasized that, in mathematical analysis the ordinary concepts of gradient, curl, and divergence are the operators applied to vectors, and in the theory of exterior forms, gradients, curls, and divergences obtained as the results of exterior differentiating (forms of the zero- first- and second degrees) are operators applied to pseudovectors (an axial vector).

These following examples are examples of exterior differential forms.

From these examples one can assure oneself that, firstly, the differential of an exterior form is also an exterior form (but with the degree greater by one), and, secondly, one can see that the components of differential forms are commutators of the coefficients of the form's differential. Thus, the differential of the first-degree form  $\omega = a_i dx^i$  can be written as  $d\omega = K_{ij} dx^i dx^j$  where  $K_{ij}$  are the components of the commutator for the form  $\omega$  that are defined as  $K_{ij} = (\partial a_j / \partial x^i - \partial a_i / \partial x^j)$ .

As pointed out, by definition an exterior differential form is a skew-symmetric tensor field. To a differential form of degree  $p$  there corresponds a skew-symmetric covariant tensor of the type  $(0, p)$ :  $T = (T_{\alpha_1 \dots \alpha_p})$ . Such a tensor may be written as

$$T = \sum_{\alpha_1 < \dots < \alpha_p} T_{\alpha_1 \dots \alpha_p} e^{\alpha_1} \circ e^{\alpha_2} \circ \dots \circ e^{\alpha_p}$$

where  $e^{\alpha_i}$  are base vectors. If the differentials of coordinates  $dx^{\alpha_i}$  are chosen as the basis, then to the skew-symmetric tensor there will correspond the expression

$$T = \sum_{\alpha_1 < \dots < \alpha_p} T_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \dots \wedge dx^{\alpha_p}$$

which is a differential form. (Since differentials of coordinates must satisfy the condition of exterior multiplication (1.2), the correspondence  $dx^\alpha \wedge dx^\beta \leftrightarrow e^\alpha e^\beta - e^\beta e^\alpha$  must be valid [3]. {The forms  $dx^{\alpha_i}$  make up the basis of the cotangent space}).

[Historically, in physical applications the first method proposed for expressing tensors, namely, by means of base vectors, was widely used. Not only skew-symmetric tensors, but all tensors can be written in that form. In the case when the metric tensor exists, this way makes it possible to go from covariant indices to contravariant ones, and vice versa. Nevertheless, a second method of expressing the skew-symmetric tensors, i.e., as exterior differential forms, has distinct advantages. (Practically all authors who work with exterior differential forms direct attention to this fact [3, 5, 6, 7, 8]). The method of presentation of the skew-symmetric tensors as differential forms extends the capabilities of the mathematical apparatus based on these tensors. Tensors are known to have been introduced as objects that are transform according to a fixed rule under transformation of coordinates. Tensors are attached to a basis that can be transformed in an arbitrary way under transition to a new coordinate map. An exterior differential form is connected with differentials of coordinates that vary according to the interior characteristics of the manifold, under translation along the manifold. Using differentials of the coordinates, instead of the base vectors, enables one to directly make use of the integration and differentiation for physical applications. Instead of differentials of the coordinates, a system of linearly-independent exterior one-forms can be chosen as the basis, and this makes the description independent of the choice of coordinate system [5, 6].]

## 1.1 Closed exterior differential forms

In mathematical formalisms and mathematical physics the closed differential forms with invariant properties appear to be of greatest practical utility.

A form is called ‘closed,’ if its differential is equal to zero:

$$d\theta^p = 0 \quad (1.4)$$

From condition (1.4) one can see that a closed form is a conserved quantity. (This means that it corresponds to a conservation law, namely, to some conserved physical quantity.)

The differential of a form is a closed form. That is

$$dd\omega = 0 \quad (1.5)$$

where  $\omega$  is an arbitrary exterior form.

A form which is the differential of some other form:

$$\theta^p = d\theta^{p-1} \quad (1.6)$$

is called an ‘exact’ form. Exact forms prove to be closed automatically

$$d\theta^p = dd\theta^{p-1} = 0 \quad (1.7)$$

This follows from the property (1.5) of the the exterior differential [5].

Here it is necessary to pay attention to the following points. In the formulas presented above it was implicitly assumed that the differential operator  $d$  is a total operator (i.e., it  $d$  acts everywhere in the vicinity of the point considered locally), and therefore it acts on the manifold of the initial dimension  $n$ . However, the differential may be internal. Such a differential acts on some structure with the dimension being less than that of the initial manifold. The structure, on which the exterior differential form may become a closed inexact form, is a pseudostructure with respect to its metric properties. {Cohomology (de Rham cohomology, singular cohomology [4, 6]), sections of cotangent bundles, integra and potential surfaces and so on, may be regarded as examples of pseudostructures. As it will be shown later, an eikonal surfaces corresponds to a pseudostructure}.

If a form is closed on a pseudostructure only, the closure condition is written as

$$d_\pi \theta^p = 0 \quad (1.8)$$

(In this case the internal differential (rather than total one) becomes equal to zero). And the pseudostructure  $\pi$  obeys the condition

$$d_\pi {}^* \theta^p = 0 \quad (1.9)$$

where  ${}^* \theta^p$  is a dual form. (For the properties of dual forms, see [6]).

[In a two-dimensional space,  $(x, y)$ , an exterior differential form can be written as  $\theta = udx + vdy$ , and the corresponding dual form is  ${}^* \theta = -vdx + udy$ .]

From conditions (1.8) and (1.9) one can see that the form closed on a pseudostructure is a conserved object, namely, this quantity is conserved on a pseudostructure. (This can also correspond to some conservation law, i.e. to a conserved object.)

An exact form is, by definition, a differential (see condition (1.6)). In this case the differential is total. A closed inexact form is a differential too; and, in this case the differential is an interior one defined on a pseudostructure. Thus, any closed form is a differential. The exact form is a total differential. The closed inexact form is an interior (on pseudostructure) differential, that is

$$\theta_\pi^p = d_\pi \theta^{p-1} \quad (1.10)$$

At this point it is worth noting that the total differential of a form closed on the pseudostructure is nonzero, that is

$$dd_\pi \omega \neq 0 \quad (1.11)$$

And so, any closed form is a differential of a form of lower degree: the total one  $\theta^p = d\theta^{p-1}$  if the form is exact, or the interior one  $\theta^p = d_\pi \theta^{p-1}$  on pseudostructure if the form is inexact. (From this it follows that the form of lower degree may correspond to a potential, and the closed form by itself may correspond to a potential force. This is an additional example showing that a closed form may have physical meaning. Here the two-fold nature of a closed form is revealed, on the one hand, as a locally conserved quantity, and on the other hand, as a potential force.)

From the conditions (1.6) and (1.10), it is possible to see that a connection between closed forms of different degrees can exist. Due to condition (1.5), for exact forms this connection couples only two forms. If the form  $\theta^p$  is exact, then there exists a connection between the forms  $\theta^p$  and  $\theta^{p-1}$ :  $\theta^p = d\theta^{p-1}$ , but the form  $\theta^{p+1} = d\theta^p$  vanishes according to condition (1.11), and the connection is broken. In this case it is assumed that the form  $\theta^{p-1}$  is not exact, since otherwise the form  $\theta^p$  would be equal to zero. There is then no connection with the form  $\theta^{p-2}$ . For closed inexact forms, for which condition (1.11) is satisfied, such connections may couple greater numbers of terms. {In differential geometry this fact is related to cohomology theory, the theory of structures}.

Similarly to the differential connection between exterior forms of sequential degrees, there is an integral connection. The relevant integral relation has the form [6]

$$\int_{c^{p+1}} d\theta^p = \int_{\partial c^{p+1}} \theta^p \quad (1.12)$$

In particular, the integral theorems by Stokes and Gauss follow from the integral relation for  $p = 1, 2$  in three-dimensional space. {From this relation one can see that the integral of the closed form over the closed curve vanishes (in the case of a smooth manifold). However, in the case of a complex manifold (for example, a not simply connected manifold, with the homology class being nonzero), an integral of a closed form (in this case the form is inexact) over a closed curve is nonzero. It may be equal to a scalar multiplied by  $2\pi$ , which in this case corresponds to, for example, a physical quantity such as charge [6]. Just such integrals are considered in the theory of residues.}

## 1.2 Properties of the closed exterior forms

The role of closed exterior forms in mathematics relates to the fact that the properties of closed exterior and dual forms, namely, invariance, covariance, conjugacy, and duality, lie at the basis of the group, structural and other invariant methods of mathematics.

### Invariant properties of closed exterior differential forms.

Since a closed form is a differential, then it is obvious that a closed form will turn out to be invariant under all transformations that conserve the differential. (The nondegenerate transformations in mathematics and mathematical physics such as the unitary, tangent, canonical, gradient, and other nondegenerate transformations are examples of such transformations that conserve the differential.)

Invariant properties of closed exterior forms explicitly or implicitly manifest themselves essentially in all invariant mathematical formalisms and formalisms of field theory, such as the Hamilton formalism, tensor calculus, group theory, quantum mechanics equations, Yang-Mills theory and others.

Covariance of a dual form is directly connected with the invariance of an exterior closed form.

The invariance property of an closed inexact exterior form and covariance of its dual form play an important role in describing invariant structures and manifolds.

#### Invariant structures

Of most significance in mathematical formalisms and mathematical physics are closed inexact exterior forms. This is due to the fact that the closed inexact exterior form and relevant dual form describe the differential-geometrical structure, which is invariant one.

From the definition of a closed inexact exterior form one can see that to this form there correspond two conditions:

- (1) condition (1.8) is the closure condition of the exterior form itself, and
- (2) condition (1.9) is that of the dual form.

Conditions (1.8) and (1.9) can be regarded as equations for a binary object that combines the pseudostructure (dual form) and the conserved quantity (the exterior differential form) defined on this pseudostructure. Such a binary object is differential - geometrical structure. (The well-known G-Structure is an example of such differential-geometrical structure.)

As it has been already pointed out, closed inexact exterior form is a differential (an interior one on the pseudostructure), and hence it remains invariant under all transforms that conserve the differential. Therefore, the relevant differential-geometrical structure also remains invariant under all transforms that conserve the differential. For the sake of convenience in the subsequent presentation such differential - geometrical structures will be called the I-Structures.

To the unique role of such invariant structures in mathematics it points the fact that the transformations conserving the differential (unitary, tangent, canonical, gradient and gauge ones) lie at the basis of many branches of mathematics, mathematical physics and field theory.

The invariant structures appear while analyzing the integrability of differential equations. Their role in the theory of differential equations relates to the fact that they correspond to generalized solutions which describe measurable physical quantities. In this case the integral surfaces with conservative quantities (like the characteristics, the characteristic surfaces, potential surfaces and so on) are invariant structures. The examples of such studying the integrability of differential equations using the skew-symmetric differential forms are presented in paper [5].

The mechanism of realization of the differential-geometrical structures and their characteristics will be described in Subsection 2.4.

### 1.3 Invariance as the result of conjugacy of elements of exterior or dual forms

Closure of exterior differential forms, and hence their invariance, results from the conjugacy of the elements of exterior or dual forms.

From the definition of an exterior differential form one can see that exterior differential forms have complex structure. The specific features of the structure of exterior forms are homogeneity with respect to the basis, skew-symmetry, the integration of terms each consisting of two objects of different nature (the algebraic nature for the form coefficients, and the geometric nature of the base components). Besides, an exterior form depends on the space dimension and on the manifold topology. The closure property of an exterior form implies that any objects, namely, elements of the exterior form, components of elements, elements of the form's differential, exterior and dual forms and others, turn out to be conjugated. It is conjugacy that leads to realization of the invariant and covariant properties of the exterior and dual forms that have great functional and applied importance. The variety of conjugate objects leads to the fact that closed forms can describe a great number of different physical and spatial structures, and this fact, once again, emphasizes the vast mathematical capabilities of the exterior differential forms.

Let us consider some types of conjugacy that make the exterior differential and dual forms closed, that is, they make these form differentials equal to zero.

As it was pointed out already, the components of an exterior form a commutator are the coefficients of the differential of this form. If the commutator of the form vanishes, the form differential vanishes too, and this indicates that the form is closed. Therefore, closure of a form may be recognized by finding whether or not the commutator of the form vanishes.

One of the types of conjugacy is that for the form coefficients.

Let us consider an exterior differential form of the first degree  $\omega = a_i dx^i$ . In this case the differential will be expressed as  $d\omega = K_{ij} dx^i dx^j$ , where  $K_{ij} = (\partial a_j / \partial x^i - \partial a_i / \partial x^j)$  are the components of the form's commutator.

It is evident that the differential may vanish if the components of commutator vanish. One can see that the components of the commutator  $K_{ij}$  may vanish if derivatives of the form's coefficients vanish. This is a trivial case. In addition, the components  $K_{ij}$  may vanish if the coefficients  $a_i$  are derivatives of some function  $f(x^i)$ , that is,  $a_i = \partial f / \partial x^i$ . In this case, the components of the commutator are equal to the difference of the mixed derivatives

$$K_{ij} = \left( \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right)$$

and therefore they vanish. One can see that those form coefficients  $a_i$ , that satisfy these conditions, are conjugated quantities (the operators of mixed differentiation turn out to be commutative).

Let us consider the case when the exterior form is written as

$$\theta = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

where  $f$  is the function of two variables  $(x, y)$ . It is evident that this form is closed because it is equal to the differential  $df$ . And for its dual form

$$*\theta = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy$$

to be closed also, it is necessary that its commutator be equal to zero

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv \Delta f = 0$$

where  $\Delta$  is the Laplace operator. As a result, the function  $f$  has to be harmonic.

Let us assume that the exterior differential form of first degree has the form  $\theta = u dx + v dy$ , where  $u$  and  $v$  are functions of two variables  $(x, y)$ . In this case, the closure condition of the form, that is, the condition under which the form commutator vanishes, takes the form

$$K = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0$$

One can see that this is one of the Cauchy-Riemann conditions for complex functions. The closure condition for the relevant dual form  $*\theta = -v dx + u dy$  is the second Cauchy-Riemann condition. {Here one can see a connection between exterior differential forms and functions of complex variables. If we consider the function  $w = u + iv$  of the complex variables  $z = x + iy$ ,  $\bar{z} = x - iy$  which satisfy the Cauchy-Riemann conditions, then to closed exterior and dual forms will correspond to this function. (The Cauchy-Riemann conditions are the conditions under which a function of complex variables does not depend on the conjugated coordinate  $\bar{z}$ ). And, to each harmonic function of complex variables there corresponds the closed exterior differential form, whose coefficients  $u$  and  $v$  are conjugated harmonic functions}.



A conjugacy, which makes an interior differential on apseudostructure equal to zero, can exist, say,  $d_\pi\theta = 0$ . Let us assume that an interior differential is a form of first degree (the form itself is that of zero degree), and it can be given in the form  $d_\pi\theta = p_x dx + p_y dy = 0$ , where  $p$  is a form of zero degree (any function). Then the closure condition of the form is

$$\frac{dx}{dy} = -\frac{p_y}{p_x} \quad (1.13)$$

This is a conjugacy of the basis and the derivatives of the form's coefficients. One can see that formula (1.13) is one of the formulas of the canonical relations [11]. The second formula of the canonical relations follows from the condition that the dual form differential vanishes. This type of conjugacy is connected with a canonical transformation. For the differential of first degree form (in this case the differential is a form of second degree), the corresponding transformation is a gradient one.

Relation (1.13) is the condition for an implicit function to exist. That is, the closed (inexact) form of zero degree is the implicit function. {This is an example of a connection between an exterior forms and analysis}.

The property of the exterior differential forms being closed on a pseudostructure points to another type of conjugacy, namely, the conjugacy of exterior and dual forms.

## Duality of the exterior differential forms

Conjugacy is connected with another characteristic property of exterior differential forms, namely, their duality; and, as will be shown later, it has fundamental physical meaning. {Conjugacy is an identical connection between two operators or mathematical objects. Duality is a concept meaning that one object carries a double meaning, or that two objects with different meanings (of different physical nature) are identically connected. If one knows any dual object, one can obtain the other object}. The conjugacy of objects of the exterior differential form generates duality of the exterior forms.

The connection between exterior and dual forms is an example of duality. An exterior form and its dual form correspond to objects of different nature: an exterior form corresponds to a physical (i.e. algebraic) quantity, and its dual form corresponds to some spatial (or pseudospacial) structure. At the same time, under conjugacy, the duality of these objects manifests itself, that is, if one form is known it is possible to find the other form. {It will be shown below that the duality between exterior and dual forms elucidates a connection between physical quantities and spatial structures (together they form a physical structure)}. Here duality is also evident in the fact that, if the degree of the exterior form equals  $p$ , the dimension of the structure equals  $N - p$ , where  $N$  is the space dimension.

Since a closed exterior form possesses invariant properties and the dual form corresponding to it possesses covariant properties, invariance of the closed ex-

terior form and the relevant covariance of its dual form, is an example of the duality of exterior differential forms.

Another example of duality of closed forms is connected with the fact that a closed form of degree  $p$  is a differential form of degree  $p - 1$ . This duality is manifested in that, on the one hand, as it was pointed out before, the closed exterior form is a conserved quantity, and on the other hand, the closed form can correspond to a potential force. (Below the physical meaning of this duality will be elucidated, and it will be shown in respect to what a closed form manifests itself as a potential force and with what the conserved physical quantity is connected).

A further manifestation of duality of exterior differential forms, that needs more attention, is the duality of the concepts of closure and of integrability of differential forms.

A form that can be presented as a differential can be called integrable, because it is possible to integrate it directly. In this context a closed form, which is a differential (total, if the form is exact, or interior, if the form is inexact), is integrable. And the closed exact form is integrable identically, whereas the closed inexact form is integrable on pseudostructure only.

The concepts of closure and integrability are not identical. Closure of a form is defined with respect to the form of degree greater by one, whereas integrability is defined with respect to a form of degree less by one. Really, a form is closed if the form's differential, which is a form of greater by one degree, equals zero. And a form referred to as integrable is a differential of some form of degree less by one. Closure and integrability are dual concepts. Namely, closure and integrability are further examples of duality among exterior differential forms.

Here it should be emphasized that duality is a property of closed forms only. In particular, nonclosure and nonintegrability are not dual concepts, and this will be revealed while analyzing the evolutionary differential forms.

Duality of closed differential forms is revealed by an availability of one or another type of conjugacy. As it will be shown below, this has physical significance. Duality is a tool that untangles the mutual connection, the mutual changeability and the transitions between different physical objects in the evolutionary processes.

## 1.4 Specific features of the mathematical apparatus of exterior differential forms

### Operators of the theory of exterior differential forms

The distinguishing properties of the mathematical apparatus of exterior differential forms were identified by Cartan [7]. Cartan aimed to build a theory, which contains concepts and operations being independent of any change of variables, both dependent and independent. To do this it was necessary to exchange partial derivatives for differentials that have an interior meaning.

In the differential calculus, derivatives are the basic elements of the mathematical apparatus. By contrast, a differential is an element of the mathematical

apparatus of the theory of exterior differential forms. It enables one to analyze the conjugacy of derivatives in various directions, which extends the potentialities of the differential calculus. (A derivative that can be considered as the conjugacy of the differentiating operator and scalar function is an analog of the exterior zero-degree differential).

The exterior differential operator,  $d$ , is an abstract generalization of the ordinary mathematical operations of gradient, curl and divergence from vector calculus [5]. If, in addition to the exterior differential, we introduce the following operators: (1)  $\delta$  for transformations that convert the form of degree  $p + 1$  into the form of degree  $p$ , (2)  $\delta'$  for cotangent transformation, (3)  $\Delta$  for the  $d\delta - \delta d$  transformation, (4)  $\Delta'$  for the  $d\delta' - \delta'd$  transformation, then in terms of these operators, that act on the exterior differential forms, one can write down the operators in field theory equations. The operator  $\delta$  corresponds to Green's operator,  $\delta'$  to the canonical transformation operator,  $\Delta$  to the d'Alembert operator in 4-dimensional space, and  $\Delta'$  corresponds to the Laplace operator [6, 8]. It can be seen that the operators of the exterior differential form theory are connected with many operators of mathematical physics.

The mathematical apparatus of exterior differential forms extends the capabilities of integral calculus. As pointed out, exterior differential forms were introduced as integrand expressions for definition of the integral invariants. The closure condition for exterior differential forms makes it possible to find integrability conditions, namely, to find the conditions under which the integral is independent of the integration curve. The connection between forms of subsequent degrees establishes the relations which under some conditions make it possible to reduce integration over some domain to that along some boundary, and vice versa (see formula (1.6)). This fact is of great importance for applications.

## Identical relations of exterior differential forms

In the theory of exterior differential forms closed forms, that possess various types of conjugacy, play a principal role. Since conjugacy represents a certain connection between two operators or mathematical objects, it is evident that relations can be used to express conjugacy mathematically. Just such relations constitute the basis of the mathematical apparatus of exterior differential forms.

At this point the following it should be emphasized. A relation is a comparison, a correlation of two objects. A relation may be an identical or nonidentical (see Section 2).

The basis of the mathematical apparatus of exterior differential forms comprises identical relations. (Below nonidentical relations will be discussed, and it will be shown that identical relations for exterior differential forms are obtained from nonidentical relations. Also it will be shown, that transitions from nonidentical relations to identical ones describe transitions of any quality into another one).

The identical relations of exterior differential forms reflect the closure conditions of the differential forms, namely, the vanishing of the form's differential

(see formulas (1.4), (1.8), (1.9)) and the conditions connecting forms of sequential degrees (see formulas (1.6), (1.10), (1.12)).

The importance of identical relations for exterior differential forms is manifested by the fact that practically in all branches of physics, mechanics, thermodynamics one encounters such identical relations.

Several kinds of identical relations can be distinguished.

1. *Relations among differential forms.*

These are relations connecting forms of sequential degrees. They correspond to formulas (1.6), (1.10). Examples of such identical relations are

- a) the Poincare invariant  $ds = -H dt + p_j dq_j$ ,
- b) the second principle of thermodynamics  $dS = (dE + p dV)/T$ ,
- c) the vital force theorem in theoretical mechanics:  $dT = X_i dx^i$  where  $X_i$  are the components of a potential force, and  $T = mV^2/2$  is the vital force,
- d) the conditions on characteristics [9] in the theory of differential equations.

The requirement that the function is an antiderivative (the integrand is a differential of a certain function) can be written in terms of such an identical relation.

The existence of a harmonic function is expressed by means of an identical relation: a harmonic function is a closed form, that is, a differential (a differential on the Riemann surface).

Identical relations among differential forms expresses the fact that each closed exterior form is a differential of some exterior form (with the degree less by one).

In general, such an identical relation can be written as

$$d\phi = \eta^p \quad (1.14)$$

In this relation the form on the right-hand side has to be a *closed*. (As will be shown below, the identical relations are satisfied only on pseudostructures. That is, an identical relation can be written as

$$d_\pi \phi = \eta_\pi^p \quad (1.14')$$

In the identical relations (1.14) on one side there is a closed form, and on other a differential of some differential form of degree the less by one.

From such identical relations one can obtain the following information.

If there is a differential of some exterior form, it means that there is a closed exterior form (which points, for example, to the availability of the structure).

On the other hand, the availability of a closed exterior form points to the availability of a differential (and this may mean that there is a potential or a state function).

From such a relation it follows that a differential is the result of conjugacy of some objects, because this differential is a closed form. Hence a potential or a state function is also connected with the conjugacy of certain objects.

2. *Integral identical relations.*

At the beginning of the paper, it was pointed out that exterior differential forms were introduced as integrands possessing the following property: they can

have integral invariants. This fact (the availability of an integral invariant) is mathematically expressed as an identical relation.

Many formulas of Newton, Leibnitz, Green, the integral relations by Stokes, Gauss-Ostrogradskii are examples of integral identical relations (see formula (1.12)).

### 3. *Tensor identical relations.*

From relations that connect exterior forms of sequential degrees, one can obtain vector and tensor identical relations that connect the operators: gradient, curl, divergence and so on.

From closure conditions on exterior and dual forms, one can obtain identical relations such as the gauge relations in electromagnetic field theory, the tensor relations between connectednesses and its derivatives in gravity theory (the symmetry of connectednesses with respect to lower indices, the Bianchi identical, the conditions imposed on the Christoffel symbols), and so on.

### 4. *Identical relations between derivatives.*

Identical relations between derivatives correspond to closure conditions on exterior and dual forms. Examples of such relations are the above presented Cauchy-Riemann conditions in the theory of complex variables, the transversality condition in the calculus of variations, the canonical relations in the Hamilton formalism, the thermodynamic relations between derivatives of thermodynamic functions [10], the condition that the derivative of implicit functions are subject to, the eikonal relations [11], and so on.

The examples presented above show that identical relations between exterior differential forms occur in various branches of mathematics and physics.

Application of identical relations, provided with a knowledge of closed forms, provides the capability (a) to find other forms that are necessary for describing physical phenomena, (b) to answer the question of whether the obtained exterior forms are closed, (c) to get information conveyed by the closed forms, and so on. They make it possible to find closed differential forms that correspond to conservation laws, invariant structures, structures of manifolds and so on. They allow potentials and state functions to be determined.

Identical relations between exterior differential forms are mathematical expressions of various kinds of conjugacy that lead to closed exterior forms. They describe the conjugacy of many objects: the form elements, components of each element, exterior and dual forms, exterior forms of various degrees, and others. Identical relations that are connected with different kinds of conjugacy (the form elements, components of each element, forms of various degrees, exterior and dual forms, and others) elucidate invariant, structural and group properties of exterior forms, which are of great importance in applications. The invariant, structural and group properties of exterior forms that have physical meaning, manifest themselves as a result of one or another kind of conjugacy. This is expressed mathematically as an identical relation.

The functional significance of identical relations for exterior differential forms lies in the fact that they can describe the conjugacy of objects of different mathematical nature. This enables one to see the internal connections between various branches of mathematics. Due to these possibilities, exterior differential

forms have wide application in various branches of mathematics. (Below a connection of exterior differential forms with different branches of mathematics will be discussed).

The mathematical apparatus of exterior differential forms is based on identical relations of great utilitarian and functional importance. However, the question of how closed exterior differential forms and identical relations are obtained, and how the process of conjugating different objects is realized, remains open.

The mathematical apparatus of evolutionary differential forms that is based on nonidentical relations enables us to answer this question. This will be shown in Section 2.

## Nondegenerate transformations

One of the fundamental methods in the theory of exterior differential forms is application of *nondegenerate* transformations (below it will be shown that *degenerate* transformations appear in the mathematical apparatus of the evolutionary forms).

In the theory of exterior differential forms nondegenerate transformations are those that conserve the differential. Unitary transformations (0-forms), the tangent and canonical transformations (1-forms), gradient and gauge transformations (2-forms) and so on are examples of such transformations. These are the gauge transformations for spinor, scalar, vector, tensor (3-form) fields.

From the descriptions of operators comprised of exterior differential forms, one can see that they are operators that execute some transformation. All these transformations are connected with the above listed nondegenerate transformations of exterior differential forms. (It is worth pointing out that just such transformations are used extensively in field theory. The field theory operators such as Green's, d'Alembert's and Laplace operators are connected with nondegenerate transformations of closed exterior differential forms.)

The possibility of applying nondegenerate transformations shows that exterior differential forms possess group properties. This extends the utilitarian potentialities of the exterior differential forms.

The significance of nondegenerate transformations consists in the fact, that they allow one to get new closed differential forms, which opens up the possibility of obtaining new structures.

Nondegenerate transformations, if applied to identical relations, enable one to obtain new identical relations and new closed exterior differential forms.

Specific features of nondegenerate transformations, and their relation to the degenerate transformations, will be discussed in Subsection 2.3 in more detail.

From the properties of exterior differential forms presented above one can see, that the properties of exterior differential forms conform to specific features of many branches of mathematics. Below we outline connections between exterior differential forms and various branches of mathematics in order to show what

role exterior differential forms play in mathematics, and to draw attention to their great potentialities.

## 1.5 Connection between exterior differential forms and various branches of mathematics

### Connection with tensors

In Section 1.1 a connection between exterior differential forms and skew-symmetrical tensors was demonstrated. More detailed information on this subject can be found, e.g., in [3,5,8].

As was pointed out in Section 1.1, the exterior differentiation operator,  $d$ , is the abstract generalization of the gradient, curl and divergence. This property elucidates a connection between vectorial, algebraic and potential fields. The property of the exterior differential form, namely, the existence of differential and integral relations between the forms of sequential degrees, allows us to classify these fields according to their exterior form degree.

### Algebraic properties of exterior differential forms

The basis of Cartan's method of exterior forms, namely, the method of analyzing systems of differential equations and manifolds, is the basis of the 'Grassmann' algebra (i.e., exterior algebra) [1]. The mathematical apparatus of exterior differential forms extends algebraic mathematical techniques. Differential forms treated as elements of algebra, allow studying manifold structure and finding manifold invariants. The group properties of exterior differential forms (the connection with the Lie groups) enable one to study the integrability of differential equations. They can constitute the basis of the invariant field theory. They establish a connection between vectorial and algebraic fields.

### Geometrical properties of exterior differential forms

Exterior differential forms elucidate the internal connection between algebra and geometry. From the definition of a closed inexact form it follows that a closed inexact form is a quantity that is conserved on a pseudostructure. That is, a closed inexact form is a conjugacy of the algebraic and geometrical approaches. The set of relations, namely, the closure condition for exterior forms,  $(d_\pi \theta^p = 0)$ , and the closure condition for dual forms,  $(d_\pi^* \theta^p = 0)$ , allow the description of conjugacy. Closed form possess algebraic (invariant) properties, and closed dual forms have geometrical (covariant) properties.

The mathematical apparatus of exterior differential forms enables one to study the elements of the interior geometry. This is a fundamental formalism in differential geometry [7]. It enables one to investigate manifold structure. Cartan developed the method of exterior forms for the purpose of investigating manifolds. It is known that in the case of integrable manifolds, the metrical and differential characteristics are consistent. Such manifolds may be regarded

as objects of the interior geometry, i.e. the possibility of studying their characteristics as the properties of the surface itself without regard to its embedding space. Cartan's structure equation is a key tool in studying manifold structures and fiber spaces [7,12].

## Theory of functions of complex variables

The residue method in the theory of analytical functions of complex variables is based on the integral theorems by Stokes and Cauchy-Poincare that allow us to replace the integral of closed form along any closed loop by the integral of this form along another closed loop that is homological to the first one [2].

As already noted, harmonic functions (differentials on Riemann surfaces) are closed exterior forms:

$$\theta = p dx + q dy, \quad d\theta = 0$$

to which there corresponds the dual form:

$$*\theta = -q dx + p dy, \quad d^*\theta = 0$$

where  $d\theta = 0$  and  $d^*\theta = 0$  are the Cauchy-Riemann relations.

## Differential equations

On the basis of the theory of exterior differential forms the methods for studying the integrability of a system of differential equations, Pfaff equations (the Frobenius theorem), and of finding integral surfaces have been developed. This problem was considered in many works concerning exterior differential forms [5, 7, 13].

An example of an application of the theory of exterior differential forms for analyzing integrability of differential equations and determining the functional properties of the solutions to these equations is presented in Subsection 1.6.

## Connection with the differential and integral calculus

As it was already pointed out, exterior differential forms were introduced for designation of integrand expressions which can form integral invariants [1].

Exterior differential forms are connected with multiple integrals (see, for example [3]).

It should be mentioned that the closure property of the form  $f(x)dx$  indicates existence of an antiderivative of the function  $f(x)$ .

Integral relations (1.12), from which the formulas by Newton, Liebnitz, Green, Gauss-Ostrogradskii, Stokes were derived, are of great importance. Such potentials as Newton's, the potentials of simple and double layers are integrals of closed inexact forms.

Exterior differential forms extend the potentialities of the differential and integral calculus.



The theory of integral calculus establishes a connection between the differential form calculus and the homology of manifolds (cohomology).

The operator  $d$  appears to be useful for expressing the integrability conditions for systems of partial differential equations.

Thus, even from this brief description of the properties and specific features of the exterior differential forms and their mathematical apparatus, one can clearly see their connection with such branches of mathematics as algebra, geometry, mathematical analysis, tensor analysis, differential geometry, differential equations, group theory, theory of transformations and so on. This is indicative of wide functional and utilitarian potentialities of exterior differential forms. Exterior differential forms enable us to see the internal connection between various branches of mathematics and physics.

Here it arises the question of how closed exterior differential forms, which possess vast potentialities, are obtained.

It appears, that closed exterior differential forms can be obtained from the evolutionary differential forms (see Section 2).

Below, an example will be presented of an application of the mathematical apparatus of exterior differential forms that demonstrates the existence of new evolutionary skew-symmetric differential forms.

## 1.6 Qualitative investigation of the functional properties of the solutions to differential equations

The investigation of the functional properties of the solutions to differential equations on the basis of the mathematical apparatus of skew-symmetric differential forms makes it possible to understand what lies at the basis of the qualitative theory of differential equations. (The presented method of investigating solutions to differential equations is not new. Such an approach was developed already by Cartan [7] in his analysis of the integrability of differential equations. Here it is featured to demonstrate the significance of new skew-symmetric differential forms.)

The role of exterior differential forms in the qualitative investigation of solutions to differential equations is a consequence of the fact that the mathematical apparatus of these forms enables one to determine the conditions for consistency of various elements of differential equations or of systems of differential equations. This enables one, for example, to define the consistency of partial derivatives in partial differential equations, the consistency of differential equations in system of differential equations, the conjugacy of the function derivatives and of derivatives of initial data for ordinary differential equations and so on. Functional properties of solutions to differential equations depend just on whether or not conjugacy conditions are satisfied.

The basic idea of the qualitative investigation of the solutions to differential equations can be clarified by the example of the first-order partial differential equation.

Let

$$F(x^i, u, p_i) = 0, \quad p_i = \partial u / \partial x^i \quad (1.15)$$

be the partial differential equation of the first order.

Let us consider the functional relation

$$du = \theta \quad (1.16)$$

where  $\theta = p_i dx^i$  (the summation over repeated indices is implied). Here  $\theta = p_i dx^i$  is a differential form of the first degree.

The specific feature of a functional relation (1.16) is that in the general case this relation turns out to be a nonidentical.

The left-hand side of this relation involves a differential, and the right-hand side includes a differential form  $\theta = p_i dx^i$ . For this relation to be an identical, the differential form  $\theta = p_i dx^i$  must be a differential as well (like the left-hand side of relation (1.16)), that is, it has to be a closed exterior differential form. To achieve this requires the commutator  $K_{ij} = \partial p_j / \partial x^i - \partial p_i / \partial x^j$  of the differential form  $\theta$ , vanish.

In the general case, from equation (1.15) it does not follow (explicitly) that the derivatives  $p_i = \partial u / \partial x^i$  satisfy the equation (and given boundary or initial conditions of the problem) and constitute a differential. Without supplementary conditions, the commutator of the differential form  $\theta$  defined as  $K_{ij} = \partial p_j / \partial x^i - \partial p_i / \partial x^j$  is not equal to zero. The form  $\theta = p_i dx^i$  proves to be unclosed and is not a differential like the left-hand side of relation (1.16). The functional relation (1.16) appears to be a nonidentical: the left-hand side of this relation is a differential, but the right-hand side is not a differential.

The nonidentity of the functional relation (1.16) points to the fact, that without additional conditions the derivatives of the initial equation do not constitute a differential. This means that the corresponding solution to the differential equation  $u$  will not be a function of  $x^i$ . It will depend on the commutator of the form  $\theta$ , that is, it will be a functional.

To obtain the solution that is the function (i.e., the derivatives of this solution made up a differential), it is necessary to add a closure condition for the form  $\theta = p_i dx^i$  and for its dual form (in the present case the functional  $F$  plays a role of the form dual to  $\theta$ ) [7]:

$$\begin{cases} dF(x^i, u, p_i) = 0 \\ d(p_i dx^i) = 0 \end{cases} \quad (1.17)$$

If we expand the differentials, we get a set of homogeneous equations with respect to  $dx^i$  and  $dp_i$  (in the  $2n$ -dimensional space – initial and tangential):

$$\begin{cases} \left( \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} p_i \right) dx^i + \frac{\partial F}{\partial p_i} dp_i = 0 \\ dp_i dx^i - dx^i dp_i = 0 \end{cases} \quad (1.18)$$

The solvability conditions for this set (i.e., vanishing of the determinant composed of the coefficients at  $dx^i, dp_i$ ) have the form:

$$\frac{dx^i}{\partial F/\partial p_i} = \frac{-dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \quad (1.19)$$

These conditions determine an integrating direction, namely, a pseudostructure, on which the form  $\theta = p_i dx^i$  turns out to be closed, i.e. it becomes a differential, so that from relation (1.16) the identical relation is deduced. If conditions (1.19), that may be called ‘integrability conditions,’ are satisfied, the derivatives constitute a differential  $\delta u = p_i dx^i = du$  (on the pseudostructure), and the solution becomes a function. Just such solutions, namely, functions on the pseudostructures formed by the integrating directions, are the so-called generalized solutions [14]. Derivatives of generalized solutions made up the exterior forms that are closed on pseudostructures.

(In this case the pseudostructure, i.e. the integral curve, and the differential  $du$ , which is a closed form, made up an I-Structure. As one can see, the realization of I-Structure, i.e. the realization of conditions (1.19) and closed form, are necessary conditions for constructions the generalized solution. The generalized solution is a solution of the equation that corresponds to the moment of realization of the conditions of degenerate transformation. The pseudostructure, on which such a minisolution is defined, is correspondingly a minisrtucture. Such minisolutions are realized in discrete manner.)

If conditions (1.19) are not satisfied, that is, the derivatives do not form a differential, then the solution that corresponds to such derivatives will depend on the differential form commutator consisting of derivatives. That means that the solution is a functional rather than a function.

Since functions that are the generalized solutions are defined only on pseudostructures, they have discontinuities in their derivatives in directions that are transverse to the pseudostructures. The order of the derivatives with discontinuities equals the degree of the exterior form. If a form of zero degree is involved in the functional relation, then the function itself, being a generalized solution, will have discontinuities.

If we find the characteristics of equation (1.15), it appears that conditions (1.19) are the equations for the characteristics [9]. That is, the characteristics are examples of the pseudostructures on which the derivatives of the differential equation consist of closed forms and the solutions prove to be the functions (generalized solutions). (The characteristic manifolds of equation (1.15) are the pseudostructures  $\pi$  on which the form  $\theta = p_i dx^i$  becomes a closed form:  $\theta_\pi = du_\pi$ ). In this case the characteristics are pseudostructures. The characteristics and relevant closed forms on characteristics made up I-Structures.

Here it is worth noting, that coordinates of the equations of characteristics are not identical to the independent coordinates of the initial manifold on which equation (1.15) is defined. The transition from the initial manifold to the characteristic manifold appears to be a *degenerate* transform, namely, the determinant of the set of equations (1.18) becomes zero. The derivatives of

equation (1.15) are transformed from the tangent manifold to the cotangent manifold. The transition from the tangent manifold, where the commutator of the form  $\theta$  is nonzero (the form is not closed, and derivatives do not form a differential), to the characteristic manifold, namely, the cotangent manifold, where the commutator becomes equal to zero (and a closed exterior form arises, i.e. the derivatives form a differential), is an example of a degenerate transform.

A partial differential equation of the first order has been analyzed, and the functional relation with the form of the first degree analogous to an evolutionary form has been considered.

Similar functional properties are possessed by solutions to all differential equations describing any processes. And, if the order of the differential equation is  $k$ , a functional relation comprising  $k$ -degree form corresponds to this equation. For ordinary differential equations the commutator is generated at the expense of the conjugacy of the derivatives of the functions desired and those of the initial data (the dependence of the solution on the initial data is governed by the commutator).

In a similar manner, one can also investigate solutions to a set of partial differential equations.

It can be shown that the solutions to equations of mathematical physics, on which no additional external conditions are imposed, are functionals. The solutions prove to be exact (the generalized solution) only under realization of additional requirements, namely, conditions on degenerate transforms: vanishing the determinants, Jacobians and so on, that define the integral surfaces. The characteristic manifolds, the envelopes of characteristics, singular points, potentials of simple and double layers, residues and others are the examples of such surfaces.

Here mention should be made of the generalized Cauchy problem when initial conditions are given on some surface. The so called "unique" solution to the Cauchy problem, when the output derivatives can be determined (that is, when the determinant built of the expressions at these derivatives is nonzero), is a functional since the derivatives obtained in this a way turn out to be non-conjugated, that is, their mixed derivatives form a nonzero commutator, and the solution depends on this commutator.

The dependence of the solution on the commutator may lead to instability of the solution. Equations that do not satisfy integrability conditions (the conditions such as, for example, characteristics, singular points, integrating factors and others) may have unstable solutions. Unstable solutions appear in the case when the additional conditions are not realized and no exact solutions (their derivatives form a differential) are formed. Thus, the solutions to the equations of the elliptic type may be unstable.

Investigation of nonidentical functional relations lies at the basis of the qualitative theory of differential equations. It is well known that the qualitative theory of differential equations is based on the analysis of unstable solutions and the integrability conditions. From the functional relation it follows that the dependence of the solution on the commutator leads to instability, and the closure conditions of skew-symmetric forms constructed by derivatives are integrability

conditions. That is, the qualitative theory of differential equations that solves the problem of unstable solutions and integrability bases on the properties of nonidentical functional relation.

Thus, application of the exterior differential forms allows one to expose the functional properties of the solutions to differential equations.

Here it is possible to see two specific features that lie beyond the scope of the exterior differential form theory: the appearance of a nonidentical relation (see, functional relation (1.16)) and a degenerate transformation. The skew-symmetric differential forms, which, in contrast to exterior forms, are defined on nonintegrable manifolds (such, for example, as the tangent manifolds of differential equations) and possess the evolutionary properties, are provided with such mathematical apparatus.

## 2 Evolutionary skew-symmetric differential forms

In this Section the evolutionary skew-symmetric differential forms are described. Such skew-symmetric forms are deduced from evolutionary differential equations and, in contrast to exterior forms, are defined on nonintegrable manifolds (such as tangent manifolds of differential equations, Lagrangian manifolds and so on). A specific feature of evolutionary forms is the fact that from the evolutionary forms the closed exterior forms are obtained.

The process of extracting closed exterior forms from evolutionary forms describes the generation of various invariant structures, conjugated objects and operators, discrete transitions and quantum jumps, forming pseudometric and metric manifolds.

A distinction of evolutionary skew-symmetric differential forms from exterior forms is connected with the properties of manifolds on which skew-symmetric forms are defined.

In the beginning of this section information on skew-symmetric differential forms and the manifolds on which skew-symmetric differential forms can be defined is presented.

### 2.1 Some properties of manifolds

Assume that on a manifold one can place a coordinate system with base vectors  $\mathbf{e}_\mu$  and define the metric forms for a manifold [15]:  $(\mathbf{e}_\mu \mathbf{e}_\nu)$ ,  $(\mathbf{e}_\mu dx^\mu)$ ,  $(d\mathbf{e}_\mu)$ .

If a metric form is closed (i.e., its commutators equal zero), then this metric is defined by  $g_{\mu\nu} = (\mathbf{e}_\mu \mathbf{e}_\nu)$  and the results of a translation over a manifold of the point  $d\mathbf{M} = (\mathbf{e}_\mu dx^\mu)$  and of the unit frame  $d\mathbf{A} = (d\mathbf{e}_\mu)$  prove to be independent of the the path of integration. Such a manifold is integrable. [On the integrability of manifolds, see[15]].

If metric forms are nonclosed (the commutators of metric forms are nonzero), this points to the fact that this manifold is nonintegrable.

Metric forms and their commutators define the metric and differential characteristics of a manifold.

Closed metric forms define a manifold structure, i.e. the internal characteristics of a manifold. And, nonclosed metric forms define the differential characteristics of a manifold. The topological properties of manifolds are connected with commutators of nonclosed metric forms. The commutators of nonclosed metric forms define the manifold differential characteristics that specify the manifold deformations: bending, torsion, rotation, twist. Thus, the final result is, that nonintegrable manifolds, i.e. the manifolds with nonclosed metric forms, are deformed manifolds.

To describe manifold differential characteristics and, correspondingly, metric form commutators, one can use connectedness [2,5,7,15].

If the components of a metric form can be expressed in terms of connectedness  $\Gamma_{\mu\nu}^\rho$  [15], the expressions  $\Gamma_{\mu\nu}^\rho$ ,  $(\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho)$  and  $R_{\nu\rho\sigma}^\mu$  are components of the commutators of metric forms of zeroth- first- and third degrees. (The commutator of the second degree metric form is written down in a more complex manner [15], and therefore it is not presented here).

As is known, a commutator of the zeroth degree metric form  $\Gamma_{\mu\nu}^\rho$  characterizes the bend, while that of the first degree form  $(\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho)$  characterizes the torsion, the commutator of the third degree metric form  $R_{\nu\rho\sigma}^\mu$  determines the curvature.

In the case of nonintegrable manifolds, the components of the metric form commutators are nonzero. In particular, the connectednesses  $\Gamma_{\mu\nu}^\rho$  are not symmetric. (For manifolds with a closed metric form of the first degree, the connectednesses are symmetric.)

Examples of nonintegrable manifolds are the tangent manifolds of differential equations that describe an arbitrary processes, Lagrangian manifolds, the manifolds constructed of trajectories of material system elements (particles), which are obtained while describing evolutionary processes in material media.

Below it will be shown that the properties of skew-symmetric forms depend on the properties of metric forms of the manifold on which these skew-symmetric forms are defined. The difference between exterior and evolutionary forms depends on the properties of the manifold's metric forms.

## 2.2 Properties of the evolutionary differential forms

As pointed out above, Cartan introduced the term “exterior differential forms” to denote skew-symmetric differential forms (differential forms with exterior multiplication). It was assumed that these differential forms are defined on manifolds which locally admit a one-to-one mapping into Euclidean subspaces [16] or into other manifolds or submanifolds of the same dimension. This means that these differential forms are defined on manifolds with *closed metric forms*. In general, the theory of exterior differential forms has been developed for differential forms defined on such manifolds.

Skew-symmetric differential forms defined on manifolds with *unclosed metric forms* have their own specific features that are beyond the scope of the modern theory of exterior differential forms. Therefore, it makes sense to introduce new terminology. In the present work we shall call them “evolutionary

differential forms”.

The term “evolutionary” is derived from the fact that differential forms defined on manifolds with unclosed metric forms possess the evolutionary properties. They are obtained from differential equations that describe any processes. The coefficients of these forms depend on the evolutionary variable, and the basis varies according to variation of the form coefficients.

Thus, exterior differential forms are defined on manifolds, submanifolds or on structures with closed metric forms. The evolutionary differential forms are forms defined on manifolds with metric forms that are unclosed and vary according to changing of the evolutionary variable.

[For the evolutionary differential forms we shall use a notation with Greek indices, but for exterior differential forms we shall use Latin indices as we have previously.]

Let us point out some properties of evolutionary forms and show what their difference with exterior differential forms consist in.

### Specific features of the evolutionary forms differential

An evolutionary form differential of degree  $p$  ( $p$ -form), as well as an exterior differential form, can be written as

$$\omega^p = \sum_{\alpha_1 \dots \alpha_p} a_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \dots \wedge dx^{\alpha_p} \quad 0 \leq p \leq n \quad (2.1)$$

where the local basis obeys the skew-symmetric condition

$$\begin{aligned} dx^\alpha \wedge dx^\alpha &= 0 \\ dx^\alpha \wedge dx^\beta &= -dx^\beta \wedge dx^\alpha \quad \alpha \neq \beta \end{aligned}$$

(summation on repeated indices is implied).

But, the differential of the evolutionary form cannot be written in a manner similar to that described above for exterior differential forms (see formula (1.3)). In an evolutionary form differential there appears an additional term connected with the fact that the basis of the form changes. An evolutionary differential form can be written as

$$d\omega^p = \sum_{\alpha_1 \dots \alpha_p} da_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_p} + \sum_{\alpha_1 \dots \alpha_p} a_{\alpha_1 \dots \alpha_p} d(dx^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_p}) \quad (2.2)$$

where the second term is connected with the differential of the basis.

(As is known, an exterior form differential does not include the differential of the basis. That is, in this case the differential of the basis is equal to zero.)

[Hereinafter a symbol of summing  $\sum$  and a symbol of exterior multiplication  $\wedge$  will be omitted. Summation over repeated indices is implied.]

The properties of a skew-symmetric form differential depend on the properties and specific features of the manifold on which skew-symmetric differential form can be defined.

The second term in the expression for the differential of skew-symmetric form connected with the differential of the basis is expressed in terms of the metric form commutator. For differential forms defined on a manifold with unclosed metric form, one has  $d(dx^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_p}) \neq 0$ . And for a manifold with a closed metric form, the following  $d(dx^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_p}) = 0$  is valid.

That is, for differential forms defined on a manifold with unclosed metric form, the second term is nonzero, whereas for differential forms defined on the manifold with closed metric form the second term vanishes.

For example, let us consider the first-degree form  $\omega = a_\alpha dx^\alpha$ . The differential of this form can be written as

$$d\omega = K_{\alpha\beta} dx^\alpha dx^\beta$$

where  $K_{\alpha\beta} = a_{\beta;\alpha} - a_{\alpha;\beta}$  are the components of the commutator of the form  $\omega$ , and  $a_{\beta;\alpha}$ ,  $a_{\alpha;\beta}$  are covariant derivatives. If we express the covariant derivatives in terms of the connection coefficients (if it is possible), then they can be written as  $a_{\beta;\alpha} = \partial a_\beta / \partial x^\alpha + \Gamma_{\beta\alpha}^\sigma a_\sigma$ , where the first term results from differentiating the form's coefficients, and the second term results from differentiating the basis. (In the Euclidean space covariant derivatives coincide with ordinary ones since, in this case, the derivatives of the basis vanish). If we substitute the expressions for covariant derivatives into the formula for commutator components, we obtain the following expression for the commutator components of the form  $\omega$

$$K_{\alpha\beta} = \left( \frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} \right) + (\Gamma_{\beta\alpha}^\sigma - \Gamma_{\alpha\beta}^\sigma) a_\sigma \quad (2.3)$$

Here the expressions  $(\Gamma_{\beta\alpha}^\sigma - \Gamma_{\alpha\beta}^\sigma)$  which entered into the second term are just the components of the commutator of the first-degree metric form. (For a non-integrable manifold the connection coefficients are nonsymmetric and the expressions  $(\Gamma_{\beta\alpha}^\sigma - \Gamma_{\alpha\beta}^\sigma)$  are not equal to zero).

That is, the corresponding metric form commutator will enter into the differential form commutator.

If we substitute the expressions (2.3) for the skew-symmetric differential form commutator into formula for  $d\omega$ , we obtain the following expression for the differential of the first degree skew-symmetric form

$$d\omega = \left( \frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} \right) dx^\alpha dx^\beta + (\Gamma_{\beta\alpha}^\sigma - \Gamma_{\alpha\beta}^\sigma) a_\sigma dx^\alpha dx^\beta$$

The second term in the expression for the differential of a skew-symmetric form is connected with the differential of the manifold metric form, which is expressed in terms of the metric form commutator.

While deriving formula (2.3) for the differential form commutator connection coefficients of a special type were used. However, a similar result can be obtained by applying a connection of arbitrary type, or by using another means of finding the differential of the base coordinates. For differential forms of any degree the metric form commutator of corresponding degree will be included in the commutator of the skew-symmetric differential form.



As it is known [2,5], the differential of an exterior differential form involves only a single term. There is no second term. This indicates that the metric form commutator vanishes. In other words, the manifold, on which the *exterior* differential form is defined, has a closed metric form.

The differential of the evolutionary differential form, which is defined on a manifold with unclosed metric forms, will contain two terms: the first term depends on the differential form coefficients and the other depends on the differential characteristics of the manifold.

Thus, the differentials and, correspondingly, the commutators of exterior and evolutionary forms are of different types. (As it will be shown below, this is precisely what determines the characteristic properties and peculiarities of evolutionary forms.)

What does this lead to?

## Non closure of the evolutionary differential forms

Commutators of *exterior* differential forms (differential forms defined on manifolds with closed metric forms) contain only one term obtained from the derivatives of the differential form's coefficients. Such a commutator may be equal to zero. That is, the differential of the exterior form may vanish. This means that the exterior differential form may be closed.

In contrast to this case, evolutionary differential forms cannot be closed.

Since metric forms of the manifold are unclosed, the second term of the differential form commutator involves the metric form commutator that is not equal to zero. Hence, the second term of the evolutionary form commutator will be nonzero. In addition, the terms of this commutator have a different nature. Such terms cannot compensate one another. For this reason, the differential form commutator proves to be nonzero. And this means that the differential form defined on the manifold with an unclosed metric form cannot be closed.

Evolutionary differential forms are defined on manifolds with unclosed metric forms, and therefore, the evolutionary forms turn out to be unclosed.

### 2.3 Specific features of the mathematical apparatus of evolutionary differential forms. Generate closed external forms

Since evolutionary differential forms are unclosed, their mathematical apparatus would not seem to possess capabilities connected with the algebraic, group, invariant and other such properties of closed exterior differential forms. However, the mathematical apparatus of evolutionary forms proves to be significantly wider than expected because of the fact that evolutionary differential forms can generate closed exterior differential forms. Thus we obtain the answer to the question of how the closed exterior forms arise.

Such capabilities of evolutionary forms are due to the fact that the mathematical apparatus of evolutionary differential forms includes unconventional

elements, such one as nonidentical relations, degenerate transformations, transition from nonintegrable manifold to integrable one.

## Nonidentical relations of evolutionary differential forms

In Section 1 it was shown that identical relations lie at the basis of the mathematical apparatus of exterior differential forms. In contrast to this, nonidentical relations lie at the basis of the mathematical apparatus of evolutionary differential forms.

[Identical relations establish exact correspondence between quantities (or objects) involved in a relation. It is possible in the case in which the quantities involved in the relation are invariants, in other words, measurable quantities. In a nonidentical relation one of the quantities is unmeasurable.]

A relation of evolutionary differential forms appears nonidentical as it involves an unclosed evolutionary form that is not invariant.

Nonidentical relations appear in descriptions of any processes. They may be written as

$$d\phi = \theta^p \quad (2.4)$$

Here  $\theta^p$  is the  $p$ -degree evolutionary form that is an unclosed nonintegrable form,  $\phi$  is some form of degree  $(p-1)$ , and the differential  $d\phi$  is a closed form of degree  $p$ .

Relation (2.4) is just of the same form as the identical relation (1.14) from the mathematical apparatus of exterior differential forms presented in Section 1. However, on the right-hand side of the identical relation (1.14) stands a closed form, whereas the form on the right-hand side of nonidentical relation (2.4) is an unclosed one. (As will be shown below, identical relations are satisfied only on pseudostructures, see (1.14')).

Relation (2.4) is an evolutionary relation as it involves an evolutionary form.

On the left-hand side of relation (2.4) stands a form differential, i.e. a closed form that is an invariant object. On the right-hand side stands an unclosed form that is not an invariant object. Such a relation cannot be identical.

One can come to relation (2.4) by means of analyzing the integrability of the partial differential equation. An equation is integrable if it can be reduced to the form  $d\psi = dU$ . However it appears that, if the equation is not subjected to an additional condition (the integrability condition), it is reduced to the form (2.4), where  $\omega$  is an unclosed form and it cannot be expressed as a differential.

An example of a nonidentical relations among differential forms is a functional relation (1.16) constructed by derivatives of the differential equation (see Subsection 1.6). In this case one has  $\phi = u$  and  $\omega = \theta$ .

Here we present another derivation of a nonidentical relation (2.4) that clarifies the physical meaning of this relation.

Let us consider this in terms of an example with a first degree differential form. A differential of a function of more than one variables can be an example of the first degree form. In this case the function itself is the exterior form of zero degree. The state function that specifies the state of a material system

can serve as an example of such a function. When the physical processes in a material system are being described, the state function may be unknown; but, its derivatives may be known. The values of the function's derivatives may be equal to some expressions that are obtained from the description of a real physical process. And, the goal is to find the state function.

Assume that  $\psi$  is the desired state function that depends on the variables  $x^\alpha$ , and also assume that its derivatives in various directions are known and equal to the quantities  $a_\alpha$ , namely:

$$\frac{\partial\psi}{\partial x^\alpha} = a_\alpha \quad (2.5)$$

Let us set up the differential expression  $(\partial\psi/\partial x^\alpha)dx^\alpha$  (again, summation over repeated indices is implied). This differential expression is equal to

$$\frac{\partial\psi}{\partial x^\alpha}dx^\alpha = a_\alpha dx^\alpha \quad (2.6)$$

Here the left side of the expression is a differential of the function  $d\psi$ , and the right side is the differential form  $\omega = a_\alpha dx^\alpha$ . Relation (2.6) can be written as

$$d\psi = \omega \quad (2.7)$$

It is evident that relation (2.7) is of the same type as (2.4) under the condition that the differential form degrees are equal to 1 (here the right side is a first degree form, and the left side is a differential of the function, i.e. of the zero-degree form, which is the first degree form as well).

Relation (2.7) is a nonidentical because the differential form  $\omega$  is an unclosed differential form. The commutator of this form is nonzero since the expressions  $a_\alpha$  for the derivatives  $(\partial\psi/\partial x^\alpha)$  are nonconjugated quantities. They are obtained from the description of an actual physical process and are unmeasurable quantities.

(Here it should be emphasized that the nonidentity of relation (2.7) does not mean that the mathematical description of the function's variation is not sufficiently accurate. The nonidentity of the relation means that the function's derivatives, whose values correspond to the real values in physical processes, cannot be consistent. While seeking a state function it is commonly assumed that its derivatives are conjugated quantities, that is, their mixed derivatives commutative. But for physical processes the expressions for these derivatives are usually obtained independently of one another. And they appear to be unmeasurable quantities, and hence they are not conjugated. Similar arguments may be also presented for the evolutionary relation (2.4).)

The nonidentical relation, which is evolutionary ones since it includes the evolutionary form, plays a regulating role in the process of generation of closed (inexact) exterior forms.

The process of obtaining closed exterior forms from evolutionary forms can only proceed under the degenerate transformation.

Since evolutionary forms are unclosed, the differential of evolutionary forms is nonzero. And the differential of closed exterior forms is equal to zero. From this it follows that the transition from the evolutionary form to the closed exterior form is only possible under the degenerate transformation, namely, under the transformation which does not conserve the differential.

The degenerate transformation can only take place under additional conditions. Such conditions can be realized in the evolutionary process described.

The nonidentical evolutionary relation, which appears to be a selfvarying relation, allows to find the conditions of such degenerate transformation.

### **Selfvariation of the evolutionary nonidentical relation**

An evolutionary nonidentical relation is selfvarying, because, firstly, it is a non-identical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, namely, the variation of any object of the relation in some process leads to a variation of another object; and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop. This process is governed by the evolutionary form commutator.

Varying an evolutionary form's coefficients leads to varying the first term of the commutator (see (2.3)). In accordance with this variation, it varies the second term, that is, also the metric form of the manifold varies. Since the metric form's commutators specifies the manifold's differential characteristics that are connected with manifold deformation (for example, the commutator of the zero degree metric form specifies the bend, that of second degree specifies various types of rotation, that of the third degree specifies the curvature), then it points to a manifold deformation. This means that it varies the evolutionary form basis. This, in turn, leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated. The processes of variation of the evolutionary form and the basis are governed by the evolutionary form's commutator, and it is realized in accord with the evolutionary relation.

As a result the evolutionary relation appears to be a selfvarying relation. Selfvariation of the evolutionary relation proceeds by exchange between the evolutionary form coefficients and manifold characteristics. (This is an exchange between physical quantities and space-time characteristics, namely, between the algebraic and geometrical characteristics).

The process of an evolutionary relation selfvariation cannot come to an end. This is indicated by the fact that both the evolutionary form commutator and the evolutionary relation involve unmeasurable quantities.

Just at such selfvariation of an evolutionary relation it can be realized the condition of degenerate transformation under which a closed (inexact) exterior form can be obtained from the evolutionary form.

## Degenerate transformations. Realizations of pseudostructures and closed exterior differential forms

A distinction of the evolutionary form from the closed exterior form consists in the fact that the evolutionary differential form is defined on a manifold with unclosed metric forms, and the closed exterior form can be defined only on a manifold with closed metric forms.

Hence, it follows that a closed exterior form can be obtained from the evolutionary form only under *degenerate transformation*, when a transition from the manifold with unclosed metric forms (whose differential is nonzero) to integrable manifold with closed metric forms (for which the differential is zero) takes place.

One can see that vanishing the differential of the manifold metric form is a condition of degenerate transformation.

For this reason a transition from an evolutionary form to a closed exterior form proceeds only when the differential or commutator of the metric form becomes equal to zero.

This can take place only discretely rather than identically. The coefficients of the commutator, or differential of the manifold metric form, as well as coefficients of the commutator or the evolutionary form differential, have different natures. Therefore, they cannot identically compensate one another (they cannot vanish identically). However, the commutators or differentials of a metric form can vanish given certain combinations of their coefficients. Such combinations (the conditions for a degenerate transformation) may be realized under selfvarying the evolutionary relation.

Since the closed metric form describes a pseudostructure, vanishing of the metric form commutator (the realization of conditions for a degenerate transformation) and emergence of a closed metric form (the dual form) points to the realization of a pseudostructure and arises a closed inexact exterior form.

It is clear that the transition from evolutionary form to closed one is only possible if the closed metric form is realized.

(The vanishing of one term of an evolutionary form commutator, namely, the metric form commutator, leads to the fact that the second term of the commutator also vanishes. This is due to the fact that the terms of an evolutionary form commutator correlate with one another. An evolutionary form commutator, and, correspondingly, the differential, vanish on a pseudostructure, and this means that there arises a closed inexact exterior form.)

Thus, if the conditions for a degenerate transformation are realized, from an unclosed evolutionary form, which the differential is nonzero  $d\theta^p \neq 0$ , one can obtain a differential form closed on a pseudostructure. The differential of this form equals zero. That is, it is realized the transition:

$$d\theta^p \neq 0 \rightarrow (\text{a degenerate transformation}) \rightarrow d_\pi \theta^p = 0, d_\pi^* \theta^p = 0$$

### Conditions of a degenerate transformation.

The Cauchy-Riemann conditions, the characteristic relations, the canonical relations, the Bianchi identities and others are examples of the conditions of

degenerate transformations (vanishing the dual form differential).

Conditions of degenerate transformation, that is, additional conditions, can be realized (under selfvariation of a nonidentical relation), for example, if there appear any symmetries of the evolutionary or dual form coefficients or their derivatives.

(At describing material systems this can be caused by an availability of any degrees of freedom of material system.)

Corresponding to the conditions of degenerate transformation there is a requirement, that certain functional expressions become equal to zero. Such functional expressions (as it was pointed above in Subsection 1.6) are Jacobians, determinants, the Poisson brackets, residues, and others.

Mathematically, a degenerate transformation is realized as a transition from one frame of reference to another (nonequivalent) frame of reference. This is a transition from the frame of reference connected with the manifold whose metric forms are unclosed to the frame of reference being connected with a pseudostructure. The first frame of reference cannot be inertial or a locally-inertial frame. The evolutionary form and nonidentical evolutionary relation are defined in the noninertial frame of reference. But the thereby obtained closed exterior form and the identical relation are obtained with respect to the locally-inertial frame of reference.

## Obtaining an identical relation from a nonidentical

In Section 1 it was shown that identical relations of closed exterior form lie at the basis of many branches of mathematics.

As one can see, the identical relations of closed exterior form follow from nonidentical relations of unclosed evolutionary form.

On the pseudostructure  $\pi$ , from evolutionary relation (2.4) it follows the relation

$$d_\pi \phi = \theta_\pi^p \quad (2.8)$$

which turns out to be an identical relation. Indeed, since the form  $\theta_\pi^p$  is closed, on the pseudostructure this form turns out to be a differential of some differential form. In other words, this form can be written as  $\theta_\pi^p = d_\pi \omega$ . Relation (2.8) is now written as  $d_\pi \phi = d_\pi \omega$ . There are differentials on the left and right sides of this relation. This means that the relation is an identical.

Thus, it is evident that under a degenerate transformation an identical relation on a pseudostructure can be obtained from an evolutionary nonidentical relation. This is due to the realization of a closed metric form, and correspondingly, to the realization of closed exterior form, and this points to the emergence of a pseudostructure and a conserved quantity on the pseudostructure. The pseudostructure with conserved quantity constitutes a differential-geometrical structure, which, as it has been shown in Subsection 1.2, is an invariant structure. (The properties of such a structure will be defined below).

Under degenerate transformation an evolutionary form differential vanishes only on a pseudostructure. It is an interior differential. The total differential

of the evolutionary form is nonzero. The evolutionary form remains unclosed, and for this reason, the original relation, which contains the evolutionary form, remains a nonidentical.

Under realization of additional new conditions, a new identical relation can be obtained. As a result, the nonidentical evolutionary relation can generate identical relations.

(It can be shown that all identical relations in the theory of exterior differential forms are obtained from nonidentical relations by applying a degenerate transformation.)

### **Connection between nondegenerate transformations of exterior forms and degenerate transformations of evolutionary forms.**

In the theory of closed exterior forms only nondegenerate transformations, which conserve the differential, are used. Degenerate transformations of evolutionary forms are transformations that do not conserve the differential. Nevertheless, these transformations are mutually connected. Degenerate transformations execute a transition from an original deformed manifold to pseudostructures. And the nondegenerate transformations execute a transition from one pseudostructure to another. As the result of degenerate transformation, a closed inexact exterior form arises from an unclosed evolutionary form, and this points to the generation of a differential-geometrical structure. But under nondegenerate transformation a transition from one closed form to another takes place, and this points to the transition from one differential-geometrical structure to another.

### **Integration of a nonidentical evolutionary relation**

Since a closed exterior form is a differential (interior or total) of a form of less by one degree, this allows us to integrate the closed form and transition to a form of degree less by one. Such transitions are possible in identical relations that connect a closed form with a differential.

It can be shown that an integration and transitions with reduction by one degree are also possible for a nonidentical relation (i.e., with an evolutionary unclosed form). But this is possible only on condition of a degenerate transformation are available.

Under degenerate transformation from the nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains a relation that can be integrated, and as a result he obtains a relation with the differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

The resulting nonidentical relation of degree  $(p - 1)$  (relation that includes the forms of the degree  $(p - 1)$ ) can be integrated once again if the corresponding degenerate transformation has been realized and the identical relation has been formed.

By sequential integrating the evolutionary relation of degree  $p$  (in the case of realization of the corresponding degenerate transformations and forming the identical relation), one can get closed (on the pseudostructure) exterior forms of degree  $k$ , where  $k$  ranges from  $p$  to 0.

In this case one can see that under such integration the closed (on the pseudostructure) exterior forms, which depend on two parameters, are obtained. These parameters are the degree of evolutionary form  $p$  (in the evolutionary relation) and the degree of created closed forms  $k$ .

In addition to these parameters, an additional parameter arises, namely, the dimension of the space.

It is known that to a closed exterior differential forms of degree  $k$  there correspond a skew-symmetric tensors of rank  $k$  and to the corresponding dual forms there is a pseudotensors of rank  $(N - k)$ , where  $N$  is the dimension of the space. Pseudostructures correspond to such tensors, but only on the space formed.

## 2.4 Functional possibilities of evolutionary forms

The main peculiarity of evolutionary forms, which may be the deciding factor for mathematics, is the fact that the evolutionary forms generate closed inexact exterior forms, whose invariant properties lie at the basis of practically all invariant mathematical and physical formalisms.

The mathematical apparatus of exterior and evolutionary forms, which basis involves nonidentical relations and degenerate transformations, can describe transitions from nonconjugate operators to conjugate ones and generation of various structures. (None of mathematical formalisms contains such possibilities.)

### Mechanism of realization of conjugated objects and operators.

To the closed exterior forms it can be assigned conjugated operators, whereas to the evolutionary forms there correspond nonconjugated operators. The transition from the evolutionary form to the closed exterior form is that from nonconjugated operators to conjugated ones. This is expressed as the transition from the nonzero differential (unclosed evolutionary form) to the differential that equals zero (closed exterior form).

Here it should be emphasized that the properties of deforming manifolds and skew-symmetric differential forms on such manifolds, namely, evolutionary forms, play a principal role in the process of conjugating.

The process of conjugating includes the following points:

1) selfvariation of nonidentical relation, namely, mutual variations of the evolutionary form coefficients (which have the algebraic nature) and of manifold characteristics (which have the geometric nature) described by nonidentical evolutionary relation, and

2) realization of the degenerate transformation.



Hence one can see that the process of conjugating is a mutual exchange between the quantities of different nature and the degenerate transformation under additional conditions. Here it should be pointed that the condition of degenerate transformation (vanishing some functional expressions like, for example, Jacobians, determinants and so on) may be realized spontaneously while selfvarying the nonidentical relation if any symmetries appear. It is possible if the system (that is described by this relation) possesses any degrees of freedom.

One can see that the process of realization of conjugated operators or objects is described by the nontraditional mathematical apparatus, namely, by nonidentical relations and degenerate transformations.

As it has been shown above, closed exterior forms appear in many mathematical formalisms. Practically all conjugated objects are explicitly or implicitly connected with closed exterior forms. And yet it can be shown that closed exterior forms are generated by evolutionary differential forms, which are skew-symmetric differential forms defined on the nonintegrable deforming varying manifolds.

[It would be noted some specific features of mathematics. One branch of mathematics deals with conjugated operators or objects (algebra, geometry, the theory of groups, differential geometry, the theory of complex variables, exterior differential forms, and so on), whereas another deals with nonconjugated objects (differential and integral calculus, differential equations, topology and so on).

There are intersections between them. If the conjugacy conditions are known, one can obtain conjugated operators by imposing the conjugacy conditions on nonconjugated operators. However, in this case the questions of how the conjugacy conditions are realized, what is the cause of their origination and how the process of conjugating develops do not solve in any branch of mathematics.

The evolutionary differential forms answer these questions. They show how the transition from nonconjugated operators to conjugated ones proceeds.]

## **Realization of differential-geometrical structures.**

The process of generation of closed inexact exterior forms describes thereby the process of origination of the differential-geometrical structure which is an invariant structure (I-Structure).

Obtaining differential-geometrical structures is a process of conjugating the objects. Such process is, firstly, a mutual exchange between the quantities of different nature (for example, between the algebraic and geometric quantities or between the physical and spatial quantities), and, secondly, the establishment of exact correspondence (conjugacy) of these objects. This process is described by selfvariation of nonidentical relation and degenerate transformation.

### **Characteristics of the differential-geometrical structures realized.**

Since the closed exterior and dual differential forms, which correspond to I-Structure arisen, were obtained from the nonidentical relation that involves the evolutionary form, it is evident that the characteristics of such structure have to be connected:

a) with those of the evolutionary form and of the deforming manifold on which this form is defined,

b) with the values of commutators of the evolutionary form and the manifold metric form, and

c) with the conditions of degenerate transformation as well.

The condition of degenerate transformation corresponds to a realization of the closed metric (dual) form and defines the pseudostructure.

Vanishing the interior commutator of the evolutionary form (on pseudostructure) corresponds to a realization of the closed (inexact) exterior form and points to emergence of conserved (invariant) quantity.

When I-Structure originates, the value of the total commutator of the evolutionary form containing two terms is nonzero. These terms define the following characteristics of I-Structures:

a) the first term of evolutionary form commutator (which is composed of the derivatives of the evolutionary form coefficients) defines the value of the discrete change of conserved quantity, that is, the quantum, which the quantity conserved on the pseudostructure undergoes at the transition from one pseudostructure to another;

b) the second term (which is composed of the derivatives of coefficients of the metric form connected with the manifold) specifies the characteristics of I-Structures, which fixes the character of the initial manifold deformation taking place before I-Structures arose. (This characteristics fixes the deformation of original manifold that proceeded in the process of originating the differential-geometrical structure and was described by selfvariation of nonidentical relation).

The discrete (quantum) change of conserved quantity proceeds in the direction that is normal to the pseudostructure. (Jumps of the derivatives normal to the potential surfaces are examples of such changes.)

(Above it has been noted that the evolutionary form and the nonidentical relation are obtained while describing the physical processes that proceed in material systems. For this reason it is evident that the characteristics of I-Structure must also be connected with the characteristics of the material system being described.)

### **Classification of differential-geometrical structures realized.**

The closed forms that correspond to I-Structures are generated by the evolutionary relation which includes the evolutionary form of  $p$  degree. Therefore, the structures originated can be classified by the parameter  $p$ .

The other parameter is the degree of closed forms  $k$  generated by the non-identical evolutionary relation.

Thus, one can see that I-structures, to which there are assigned the closed (on the pseudostructure) exterior forms, can depend on two parameters. These parameters are the degree of evolutionary form  $p$  (in the evolutionary relation) and the degree of created closed forms  $k$ .

In addition to these parameters, another parameter appears, namely, the dimension of space. If the evolutionary relation generates the closed forms of degrees  $k$ , to them there are assigned the pseudostructures of dimensions  $(N-k)$ , where  $N$  is the space dimension.

## Forming pseudometric and metric manifolds.

At this point it should be noted that at every stage of the evolutionary process it is realized only one element of pseudostructure, namely, a certain mini-pseudostructure.

While varying the evolutionary variable the mini-pseudostructures form the pseudostructure. (The example of mini-pseudostructure is element of wave front. The element of wave front made up the pseudostructure at its motion.)

Manifolds with closed metric forms are formed by pseudostructures. They are obtained from the deforming manifolds with unclosed metric forms. In this case the initial deforming manifold (on which the evolutionary form is defined) and the manifold with closed metric forms originated (on which the closed exterior form is defined) are different objects.

It takes place the transition from the initial (deforming) manifold with unclosed metric form to the pseudostructure, namely, to the manifold with closed metric forms created. Mathematically this transition (the degenerate transformation) proceeds as *a transition from one frame of reference to another, nonequivalent, frame of reference.*

The pseudostructures, on which the closed *inexact* forms are defined, form the pseudomanifolds.

To the transition from pseudomanifolds to metric space it is assigned the transition from closed *inexact* differential forms to *exact* exterior differential forms.

It was shown above that the evolutionary relation of degree  $p$  can generate (with using the degenerate transformations) closed forms of degrees  $0, \dots, p$ . While generating closed forms of sequential degrees  $k = p, k = p - 1, \dots, k = 0$  the pseudostructures of dimensions  $(n + 1 - k)$  are obtained. As a result of transition to the exact closed form of zero degree the metric structure of the dimension  $n + 1$  is obtained.

Sections of the cotangent bundles (Yang-Mills fields), cohomologies by de Rham, singular cohomologies, pseudo-Riemannian and pseudo-Euclidean spaces, and others are examples of the pseudostructures and spaces that are formed by pseudostructures. Euclidean and Riemannian spaces are examples of metric manifolds that are obtained when changing to the exact forms. Here it should be noted that the examples of pseudometric spaces are potential surfaces (surfaces of a simple layer, a double layer and so on). In these cases the type of potential surfaces is connected with the above listed parameters.

Conserved quantities (closed exterior inexact forms) defined on pseudomanifolds (closed dual forms) constitute some fields. (The physical fields are the examples of such fields.) The fields of conserved quantities are formed from closed exterior forms at the same time when the manifolds are created from the pseudostructures.

Since the closed metric form is dual with respect to some closed exterior differential form, the metric forms cannot become closed by themselves, independently of the exterior differential form. This proves that the manifolds with closed metric forms are connected with the closed exterior differential forms.

This indicates that the fields of conserved quantities are formed from closed exterior forms at the same time when the manifolds are created from the pseudostructures. The specific feature of manifolds with closed metric forms that have been formed is that they can carry some information.

One can see that the evolutionary forms possess the properties, which enable one to describe the evolutionary processes, namely, the processes of generating the differential-geometrical structures and forming manifolds. In other mathematical formalisms there are no such possibilities that the mathematical apparatus of evolutionary and exterior skew-symmetrical forms possesses.

## Summary

It is shown that the skew-symmetric differential forms play a unique role in mathematics.

The invariant properties of exterior skew-symmetric differential forms lie at the basis of practically all invariant mathematical.

The unique role of evolutionary skew-symmetric differential forms, which were outlined in present work, relates to the fact that they generate the closed exterior forms possessing invariant properties.

Due their properties and peculiarities the closed exterior forms and evolutionary forms enable one to see the internal connection between various branches of mathematics.

Many foundations of the mathematical apparatus of evolutionary forms may occur to be of great importance for development of mathematics. The nonidentical relations, degenerate transformations, transitions from nonidentical relations to identical ones, transitions from one frame of reference to another (nonequivalent) frame, the generation of closed inexact exterior forms and invariant structures, forming fields and manifolds, the transitions between closed inexact exterior differential forms and exact forms and other phenomena may find many applications in such branches of mathematics as the qualitative theory of differential and integral equations, differential geometry and topology, the theory of functions, the theory of series, the theory of numbers, and others.

The evolutionary skew-symmetric differential forms may become a new branch in mathematics. They possess the possibilities that are contained in none of mathematical formalisms.

In the following work it will be shown a unique role of skew-symmetric differential forms in mathematical physics and field theory. Such role of skew-symmetric differential forms is due to the fact that they describe the properties of conservation laws.

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